Chapter 7: Trigonometric Equations and Identities

In the last two chapters we have used basic definitions and relationships to simplify trigonometric expressions and equations. In this chapter we will look at more complex relationships that allow us to consider combining and composing equations. By conducting a deeper study of the trigonometric identities we can learn to simplify expressions allowing us to solve more interesting applications by reducing them into terms we have studied.

Section 7.1 Solving Trigonometric Equations with Identities

In the last chapter, we solved basic trigonometric equations. In this section, we explore the techniques needed to solve more complex trig equations.

Building off of what we already know makes this a much easier task.
Consider the function $f(x) = 2x^2 + x$. If you were asked to solve $f(x) = 0$, it would be an algebraic task:

$2x^2 + x = 0$ Factor
$x(2x+1) = 0$ Giving solutions
$x = 0$ or $x = -1/2$

Similarly, for $g(t) = \sin t$, if we asked you to solve $g(t) = 0$, you can solve this using unit circle values.
$\sin(t) = 0$ for $t = 0, \pi, 2\pi$ and so on.

Using these same concepts, we consider the composition of these two functions:
$f(g(t)) = 2(\sin t)^2 + (\sin t) = 2\sin^2(t) + \sin(t)$

This creates an equation that is a polynomial trig function. With these types of functions, we use algebraic techniques like factoring, the quadratic formula, and trigonometric identities to break the equation down to equations that are easier to work with.

As a reminder, here are the trigonometric identities that we have learned so far:

Identities

Pythagorean Identities

$\cos^2(t)+\sin^2(t)=1$ $\quad$ $1+\cot^2(t)=\csc^2(t)$ $\quad$ $1+\tan^2(t)=\sec^2(t)$

Negative Angle Identities

$\sin(-t)=-\sin(t)$ $\quad$ $\cos(-t)=\cos(t)$ $\quad$ $\tan(-t)=-\tan(t)$

$\csc(-t)=-\csc(t)$ $\quad$ $\sec(-t)=\sec(t)$ $\quad$ $\cot(-t)=-\cot(t)$

Reciprocal Identities

$\sec(t)=\dfrac{1}{\cos(t)}$ $\quad$ $\csc(t)=\dfrac{1}{\sin(t)}$ $\quad$ $\tan(t)=\dfrac{\sin(t)}{\cos(t)}$ $\quad$ $\cot(t)=\dfrac{1}{\tan(t)}$

Example 1

Solve $2\sin^2(t)+\sin(t)=0$ for all solutions $0\le t<2\pi$

This equation is quadratic in sine, due to the sine squared term. As with all quadratics, we can approach this by factoring or the quadratic formula. This equation factors nicely, so we proceed by factoring out the common factor of sin(t).

$\sin(t)\left(2\sin(t)+1\right)=0$

Using the zero product theorem, we know that this product will be equal to zero if either factor is equal to zero, allowing us to break this equation into two cases:

$\sin(t)=0$ or $2\sin(t)+1=0$

We can solve each of these equations independently

$\sin(t)=0$ $\qquad$ From our knowledge of special angles

$t=0$ or $t=\pi$

$2\sin(t)+1=0$

$\sin(t)=-\dfrac{1}{2}$ $\qquad$ Again from our knowledge of special angles

$t=\dfrac{7\pi}{6}$ or $t=\dfrac{11\pi}{6}$

Altogether, this gives us four solutions to the equation on $0\le t<2\pi$:

$t=0,\pi,\dfrac{7\pi}{6},\dfrac{11\pi}{6}$

Example 2

Solve $3\sec^2(t) - 5\sec(t) - 2 = 0$ for all solutions $0 \le t < 2\pi$

Since the left side of this equation is quadratic in secant, we can try to factor it, and hope it factors nicely.

If it is easier to for you to consider factoring without the trig function present, consider using a substitution $u = \sec(t)$, leaving $3u^2 - 5u - 2 = 0$, and then try to factor:

$3u^2 - 5u - 2 = (3u + 1)(u - 2)$

Undoing the substitution,

$(3\sec(t) + 1)(\sec(t) - 2) = 0$

Since we have a product equal to zero, we break it into the two cases and solve each separately.

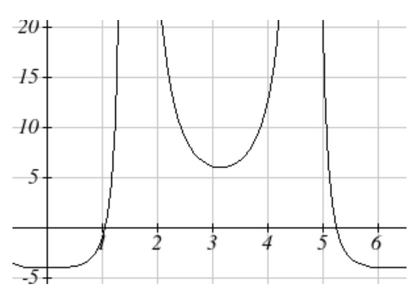
$3\sec(t) + 1 = 0$	Isolate the secant
$\sec(t) = -\frac{1}{3}$	Rewrite as a cosine
$\frac{1}{\cos(t)} = -\frac{1}{3}$	Invert both sides
$\cos(t) = -3$	

Since the cosine has a range of [-1, 1], the cosine will never take on an output of -3. There are no solutions to this part of the equation.

Continuing with the second part,

$\sec(t) - 2 = 0$	Isolate the secant
$\sec(t) = 2$	Rewrite as a cosine
$\frac{1}{\sec(t)} = 2$	Invert both sides
$\cos(t) = \frac{1}{2}$	This gives two solutions
$t = \frac{\pi}{3}$ or $t = \frac{5\pi}{3}$	

These are the only two solutions on the interval. By utilizing technology to graph $f(t) = 3\sec^2(t) - 5\sec(t) - 2$, a look at a graph confirms there are only two zeros for this function, which assures us that we didn't miss anything.



Try it Now

1. Solve $2\sin^2(t)+3\sin(t)+1=0$ for all solutions $0 \le t < 2\pi$

When solving some trigonometric equations, it becomes necessary to rewrite the equation first using trigonometric identities. One of the most common is the Pythagorean identity, $\sin^2(\theta)+\cos^2(\theta)=1$ which allows you to rewrite $\sin^2(\theta)$ in terms of $\cos^2(\theta)$ or vice versa,

$$\sin^2(\theta)=1-\cos^2(\theta)$$
$$\cos^2(\theta)=1-\sin^2(\theta)$$

This identity becomes very useful whenever an equation involves a combination of sine and cosine functions, and at least one of them is quadratic

Example 3

Solve $2\sin^2(t)-\cos(t)=1$ for all solutions $0 \le t < 2\pi$

Since this equation has a mix of sine and cosine functions, it becomes more complex to solve. It is usually easier to work with an equation involving only one trig function. This is where we can use the Pythagorean identity.

$2\sin^2(t)-\cos(t)=1$	Using $\sin^2(\theta)=1-\cos^2(\theta)$
$2\left(1-\cos^2(t)\right)-\cos(t)=1$	Distributing the 2
$2-2\cos^2(t)-\cos(t)=1$	

Since this is now quadratic in cosine, we rearranging the equation to set it equal to zero and factor.

$-2\cos^2(t)-\cos(t)+1=0$	Multiply by -1 to simplify the factoring
$2\cos^2(t)+\cos(t)-1=0$	Factor
$\left(2\cos(t)-1\right)\left(\cos(t)+1\right)=0$	

This product will be zero if either factor is zero, so we can break this into two separate equations and solve each independently.

$2\cos(t)-1=0$	or	$\cos(t)+1=0$
$\cos(t)=\dfrac{1}{2}$	or	$\cos(t)=-1$
$t=\dfrac{\pi}{3}$ or $t=\dfrac{5\pi}{3}$	or	$t=\pi$

Try it Now

2. Solve $2\sin^2(t) = 3\cos(t)$ for all solutions $0 \le t < 2\pi$

In addition to the Pythagorean identity, it is often necessary to rewrite the tangent, secant, cosecant, and cotangent as part of solving an equation.

Example 4

Solve $\tan(x) = 3\sin(x)$ for all solutions $0 \le x < 2\pi$

With a combination of tangent and sine, we might try rewriting tangent

$\tan(x) = 3\sin(x)$

$\dfrac{\sin(x)}{\cos(x)} = 3\sin(x)$ Multiplying both sides by cosine

$\sin(x) = 3\sin(x)\cos(x)$

At this point, you may be tempted to divide both sides of the equation by sin(x). **Resist the urge**. When we divide both sides of an equation by a quantity, we are assuming the quantity is never zero. In this case, when $\sin(x) = 0$ the equation is satisfied, so we'd lose those solutions if we divided by the sine.

To avoid this problem, we can rearrange the equation to be equal to zero[1].

$\sin(x) - 3\sin(x)\cos(x) = 0$ Factoring out $\sin(x)$ from both parts

$\sin(x)\left(1 - 3\cos(x)\right) = 0$

From here, we can see we get solutions when $\sin(x) = 0$ or $1 - 3\cos(x) = 0$.

Using our knowledge of the special angles of the unit circle

$\sin(x) = 0$ when $x = 0$ or $x = \pi$.

For the second equation, we will need the inverse cosine.

$1 - 3\cos(x) = 0$

$\cos(x) = \dfrac{1}{3}$ Using our calculator or technology

$x = \cos^{-1}\left(\dfrac{1}{3}\right) \approx 1.231$ Using symmetry to find a second solution

$x = 2\pi - 1.231 = 5.052$

We have four solutions on $0 \le x < 2\pi$

$x = 0, 1.231, \pi, 5.052$

[1] You technically *can* divide by $\sin(x)$ as long as you separately consider the case where $\sin(x) = 0$. Since it is easy to forget this step, the factoring approach used in the example is recommended.

Try it Now

3. Solve $\sec(\theta) = 2\cos(\theta)$ for the first four positive solutions.

Example 5

Solve $\frac{4}{\sec^2(\theta)} + 3\cos(\theta) = 2\cot(\theta)\tan(\theta)$ for all solutions $0 \le \theta < 2\pi$

$\frac{4}{\sec^2(\theta)} + 3\cos(\theta) = 2\cot(\theta)\tan(\theta)$ Using the reciprocal identities

$4\cos^2(\theta) + 3\cos(\theta) = 2\frac{1}{\tan(\theta)}\tan(\theta)$ Simplifying

$4\cos^2(\theta) + 3\cos(\theta) = 2$ Subtracting 2 from each side

$4\cos^2(\theta) + 3\cos(\theta) - 2 = 0$

This does not appear to factor nicely so we use the quadratic formula, remembering that we are solving for cos(θ).

$$\cos(\theta) = \frac{-3 \pm \sqrt{3^2 - 4(4)(-2)}}{2(4)} = \frac{-3 \pm \sqrt{41}}{8}$$

Using the negative square root first,

$$\cos(\theta) = \frac{-3 - \sqrt{41}}{8} = -1.175$$

This has no solutions, since the cosine can't be less than -1.

Using the positive square root,

$$\cos(\theta) = \frac{-3 + \sqrt{41}}{8} = 0.425$$

$\theta = \cos^{-1}(0.425) = 1.131$ By symmetry, a second solution can be found

$\theta = 2\pi - 1.131 = 5.152$

Important Topics of This Section

Review of Trig Identities

Solving Trig Equations

- By Factoring
- Using the Quadratic Formula
- Utilizing Trig Identities to simplify

Try it Now Answers

1. $t = \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$ on the interval $0 \le t < 2\pi$
2. $t = \frac{\pi}{3}, \frac{5\pi}{3}$ on the interval $0 \le t < 2\pi$
3. $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

Section 7.2 Addition and Subtraction Identities

In this section, we begin expanding our repertoire of trigonometric identities.

Identities

The sum and difference identities

$\cos(\alpha-\beta)=\cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta)$

$\cos(\alpha+\beta)=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)$

$\sin(\alpha+\beta)=\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)$

$\sin(\alpha-\beta)=\sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta)$

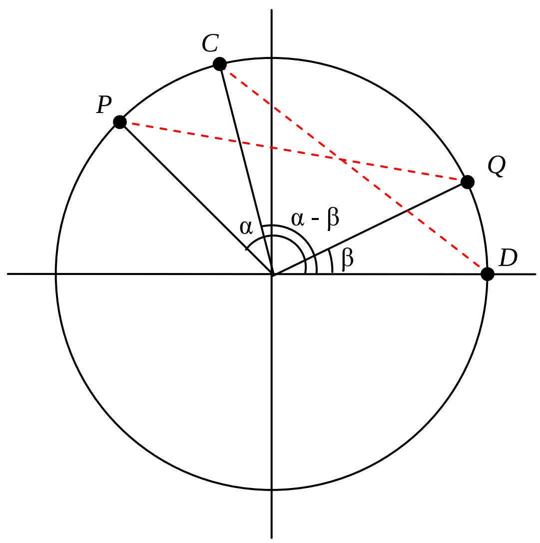
We will prove the difference of angles identity for cosine. The rest of the identities can be derived from this one.

<u>Proof of the difference of angles identity for cosine</u>

Consider two points on a unit circle:

P at an angle of α with coordinates $(\cos(\alpha),\sin(\alpha))$

Q at an angle of β with coordinates $(\cos(\beta),\sin(\beta))$



Notice the angle between these two points is α – β. Label third and fourth points:

C at an angle of α – β, with coordinates $(\cos(\alpha-\beta),\sin(\alpha-\beta))$

D at the point (1, 0)

Notice that the distance from *C* to *D* is the same as the distance from *P* to *Q*.

Using the distance formula to find the distance from *P* to *Q* is

$$\sqrt{(\cos(\alpha)-\cos(\beta))^2+(\sin(\alpha)-\sin(\beta))^2}$$

Expanding this

$$\sqrt{\cos^2(\alpha)-2\cos(\alpha)\cos(\beta)+\cos^2(\beta)+\sin^2(\alpha)-2\sin(\alpha)\sin(\beta)+\sin^2(\beta)}$$

Applying the Pythagorean Theorem and simplifying

$$\sqrt{2-2\cos(\alpha)\cos(\beta)-2\sin(\alpha)\sin(\beta)}$$

Similarly, using the distance formula to find the distance from *C* to *D*

$$\sqrt{(\cos(\alpha-\beta)-1)^2+(\sin(\alpha-\beta)-0)^2}$$

Expanding this

$$\sqrt{\cos^2(\alpha-\beta)-2\cos(\alpha-\beta)+1+\sin^2(\alpha-\beta)}$$

Applying the Pythagorean Theorem and simplifying

$$\sqrt{-2\cos(\alpha-\beta)+2}$$

Since the two distances are the same we set these two formulas equal to each other and simplify

$$\sqrt{2-2\cos(\alpha)\cos(\beta)-2\sin(\alpha)\sin(\beta)}=\sqrt{-2\cos(\alpha-\beta)+2}$$
$$2-2\cos(\alpha)\cos(\beta)-2\sin(\alpha)\sin(\beta)=-2\cos(\alpha-\beta)+2$$
$$\cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta)=\cos(\alpha-\beta)$$

Establishing the identity

Try it Now

1. By writing $\cos(\alpha+\beta)$ as $\cos(\alpha-(-\beta))$, show the sum of angles identity for cosine follows from the difference of angles identity proven above.

The sum and difference of angles identities are often used to rewrite expressions in other forms, or to rewrite an angle in terms of simpler angles.

Example 1

Find the exact value of $\cos(75°)$

Since $75° = 30° + 45°$, we can evaluate $\cos(75°)$

$\cos(75°)=\cos(30°+45°)$ — Apply the cosine sum of angles identity

$=\cos(30°)\cos(45°)-\sin(30°)\sin(45°)$ — Evaluate

$=\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}-\frac{1}{2}\cdot\frac{\sqrt{2}}{2}$ — Simply

$=\frac{\sqrt{6}-\sqrt{2}}{4}$

Try it Now

2. Find the exact value of $\sin\left(\frac{\pi}{12}\right)$

Example 2

Rewrite $\sin\left(x-\frac{\pi}{4}\right)$ in terms of sin(x) and cos(x).

$\sin\left(x-\frac{\pi}{4}\right)$ Use the difference of angles identity for sine

$=\sin(x)\cos\left(\frac{\pi}{4}\right)-\cos(x)\sin\left(\frac{\pi}{4}\right)$ Evaluate the cosine and sine and rearrange

$=\frac{\sqrt{2}}{2}\sin(x)-\frac{\sqrt{2}}{2}\cos(x)$

Additional, these identities can be used to simply expressions or prove new identities

Example 3

Prove $\dfrac{\sin(a+b)}{\sin(a-b)}=\dfrac{\tan(a)+\tan(b)}{\tan(a)-\tan(b)}$

As with any identity, we need to first decide which side to begin with. Since the left side involves sum and difference of angles, we might start there

$\dfrac{\sin(a+b)}{\sin(a-b)}$ Apply the sum and difference of angle identities

$=\dfrac{\sin(a)\cos(b)+\cos(a)\sin(b)}{\sin(a)\cos(b)-\cos(a)\sin(b)}$

Since it is not immediately obvious how to proceed, we might start on the other side, and see if the path is more apparent.

$\dfrac{\tan(a)+\tan(b)}{\tan(a)-\tan(b)}$ Rewriting the tangents using the tangent identity

$=\dfrac{\dfrac{\sin(a)}{\cos(a)}+\dfrac{\sin(b)}{\cos(b)}}{\dfrac{\sin(a)}{\cos(a)}-\dfrac{\sin(b)}{\cos(b)}}$ Multiplying the top and bottom by cos(a)cos(b)

$=\dfrac{\left(\dfrac{\sin(a)}{\cos(a)}+\dfrac{\sin(b)}{\cos(b)}\right)\cos(a)\cos(b)}{\left(\dfrac{\sin(a)}{\cos(a)}-\dfrac{\sin(b)}{\cos(b)}\right)\cos(a)\cos(b)}$ Distributing and simplifying

$= \dfrac{\sin(a)\cos(a)+\sin(b)\cos(b)}{\sin(a)\cos(a)-\sin(b)\cos(b)}$ From above, we recognize this

$= \dfrac{\sin(a+b)}{\sin(a-b)}$ Establishing the identity

These identities can also be used for solving equations.

Example 4

Solve $\sin(x)\sin(2x)+\cos(x)\cos(2x)=\dfrac{\sqrt{3}}{2}$.

By recognizing the left side of the equation as the result of the difference of angles identity for cosine, we can simplify the equation

$\sin(x)\sin(2x)+\cos(x)\cos(2x)=\dfrac{\sqrt{3}}{2}$ Apply the difference of angles identity

$\cos(x-2x)=\dfrac{\sqrt{3}}{2}$

$\cos(-x)=\dfrac{\sqrt{3}}{2}$ Use the negative angle identity

$\cos(x)=\dfrac{\sqrt{3}}{2}$

Since this is a cosine value we recognize from the unit circle we can quickly write the answers:

$x=\dfrac{\pi}{3}+2\pi k$

$x=\dfrac{5\pi}{3}+2\pi k$, where k is an integer

Combining Waves of Equal Period

Notice that a sinusoidal function of the form $f(x)=A\sin(Bx+C)$ can be rewritten using the sum of angles identity.

Example 5

Rewrite $f(x)=4\sin\left(3x+\dfrac{\pi}{3}\right)$ as a sum of sine and cosine

Using the sum of angles identity

$$4\sin\left(3x+\frac{\pi}{3}\right)$$

$$=4\left(\sin(3x)\cos\left(\frac{\pi}{3}\right)+\cos(3x)\sin\left(\frac{\pi}{3}\right)\right)$$ Evaluate the sine and cosine

$$=4\left(\sin(3x)\cdot\frac{1}{2}+\cos(3x)\cdot\frac{\sqrt{3}}{2}\right)$$ Distribute and simplify

$$=2\sin(3x)+2\sqrt{3}\cos(3x)$$

Notice that the result is a stretch of the sine added to a different stretch of the cosine, but both have the same horizontal compression which results in the same period.

We might ask now whether this process can be reversed – can a combination of a sine and cosine of the same period be written as a single sinusoidal function? To explore this, we will look in general at the procedure used in the example above.

$$f(x)=A\sin(Bx+C)$$ Use the sum of angles identity

$$=A(\sin(Bx)\cos(C)+\cos(Bx)\sin(C))$$ Distribute the *A*

$$=A\sin(Bx)\cos(C)+A\cos(Bx)\sin(C)$$ Rearrange the terms a bit

$$=A\cos(C)\sin(Bx)+A\sin(C)\cos(Bx)$$

Based on this result, if we have an expression of the form $m\sin(Bx)+n\cos(Bx)$, we could rewrite it as a single sinusoidal function if we can find values *A* and *C* so that $m\sin(Bx)+n\cos(Bx) = A\cos(C)\sin(Bx)+A\sin(C)\cos(Bx)$, which will require that:

$$m = A\cos(C) \qquad \frac{m}{A}=\cos(C)$$
$$n = A\sin(C) \qquad \frac{n}{A}=\sin(C)$$
which can be rewritten as

To find *A*,

$$m^2+n^2 = (A\cos(C))^2+(A\sin(C))^2$$

$$=A^2\cos^2(C)+A^2\sin^2(C)$$

$$=A^2(\cos^2(C)+\sin^2(C))$$ Apply the Pythagorean Identity and simplify

$$=A^2$$

Definition

To rewrite $m\sin(Bx)+n\cos(Bx)$ as $A\sin(Bx+C)$

$$A^2=m^2+n^2,\ \cos(C)=\frac{m}{A},\text{ and }\sin(C)=\frac{n}{A}$$

We can use either of the last two equations to solve for possible values of C. Since there will usually be two possible solutions, we will need to look at both to determine what quadrant C is in and determine which solution for C satisfies both equations.

Example 6

Rewrite $4\sqrt{3}\sin(2x) - 4\cos(2x)$ as a single sinusoidal function

Using the formulas above, $A^2 = \left(4\sqrt{3}\right)^2 + (-4)^2 = 16 \cdot 3 + 16 = 64$, so $A = 8$.
Solving for C,

$$\cos(C) = \frac{4\sqrt{3}}{8} = \frac{\sqrt{3}}{2}, \text{ so } C = \frac{\pi}{6} \text{ or } C = \frac{11\pi}{6}.$$

However, since $\sin(C) = \dfrac{-4}{8} = -\dfrac{1}{2}$, the angle that works for both is $C = \dfrac{11\pi}{6}$

Combining these results gives us the expression

$$8\sin\left(2x + \frac{11\pi}{6}\right)$$

Try it Now

3. Rewrite $-3\sqrt{2}\sin(5x) + 3\sqrt{2}\cos(5x)$ as a single sinusoidal function

Rewriting a combination of sine and cosine of equal periods as a single sinusoidal function provides an approach for solving some equations.

Example 7

Solve $3\sin(2x) + 4\cos(2x) = 1$ for two positive solutions.

To approach this, since the sine and cosine have the same period, we can rewrite them as a single sinusoidal function.

$A^2 = (3)^2 + (4)^2 = 25$, so $A = 5$

$$\cos(C) = \frac{3}{5}, \text{ so } C = \cos^{-1}\left(\frac{3}{5}\right) \approx 0.927 \text{ or } C = 2\pi - 0.927 = 5.356$$

Since $\sin(C) = \dfrac{4}{5}$, a positive value, we need the angle in the first quadrant, $C = 0.927$.

Using this, our equation becomes

$5\sin(2x+0.927)=1$	Divide by 5
$\sin(2x+0.927)=\frac{1}{5}$	Make the substitution $u = 2x + 0.927$
$\sin(u)=\frac{1}{5}$	The inverse gives a first solution
$u=\sin^{-1}\left(\frac{1}{5}\right)\approx 0.201$	By symmetry, the second solution is
$u=\pi-0.201=2.940$	

Undoing the substitution, we can find two positive solutions for x.

$2x+0.927=0.201$	or	$2x+0.927=2.940$
$2x=-0.726$	or	$2x=2.013$
$x=-0.363$	or	$x=1.007$

The Product to Sum and Sum to Product Identities

Identities

The Product to Sum Identities

$$\sin(\alpha)\cos(\beta)=\frac{1}{2}\left(\sin(\alpha+\beta)+\sin(\alpha-\beta)\right)$$

$$\sin(\alpha)\sin(\beta)=\frac{1}{2}\left(\cos(\alpha-\beta)-\cos(\alpha+\beta)\right)$$

$$\cos(\alpha)\cos(\beta)=\frac{1}{2}\left(\cos(\alpha+\beta)+\cos(\alpha-\beta)\right)$$

We will prove the first of these, using the sum and difference of angles identities from the beginning of the section. The proofs of the other two identities are similar and are left as an exercise.

Proof of the product to sum identity for $\sin(\alpha)\cos(\beta)$

Recall the sum and difference of angles identities from earlier

$$\sin(\alpha+\beta)=\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)$$

$$\sin(\alpha-\beta)=\sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta)$$

Adding these two equations, we obtain

$$\sin(\alpha+\beta)+\sin(\alpha-\beta)=2\sin(\alpha)\cos(\beta)$$

Dividing by 2, we establish the identity

$$\sin(\alpha)\cos(\beta)=\frac{1}{2}\left(\sin(\alpha+\beta)+\sin(\alpha-\beta)\right)$$

Example 8

Write $\sin(2t)\sin(4t)$ as a sum or difference.

Using the product to sum identity for a product of sines

$$\sin(2t)\sin(4t) = \frac{1}{2}\left(\cos(2t-4t) - \cos(2t+4t)\right)$$

$$= \frac{1}{2}\left(\cos(-2t) - \cos(6t)\right)$$ If desired, apply the negative angle identity

$$= \frac{1}{2}\left(\cos(2t) - \cos(6t)\right)$$ Distribute

$$= \frac{1}{2}\cos(2t) - \frac{1}{2}\cos(6t)$$

Try it Now

4. Evaluate $\cos\left(\frac{11\pi}{12}\right)\cos\left(\frac{\pi}{12}\right)$

Identities

The Sum to Product Identities

$$\sin(u) + \sin(v) = 2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)$$

$$\sin(u) - \sin(v) = 2\sin\left(\frac{u-v}{2}\right)\cos\left(\frac{u+v}{2}\right)$$

$$\cos(u) + \cos(v) = 2\cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)$$

$$\cos(u) - \cos(v) = -2\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right)$$

We will again prove one of these and leave the rest as an exercise.

Proof of the sum to product identity for sine functions

We begin with the product to sum identity

$$\sin(\alpha)\cos(\beta) = \frac{1}{2}\left(\sin(\alpha+\beta) + \sin(\alpha-\beta)\right)$$

We define two new variables:

$u = \alpha + \beta$

$v = \alpha - \beta$

Adding these equations yields $u+v=2\alpha$, giving $\alpha=\dfrac{u+v}{2}$

Subtracting the equations yields $u-v=2\beta$, or $\beta=\dfrac{u-v}{2}$

Substituting these expressions into the product to sum identity above,

$\sin\left(\dfrac{u+v}{2}\right)\cos\left(\dfrac{u-v}{2}\right)=\dfrac{1}{2}(\sin(u)+\sin(v))$ Multiply by 2 on both sides

$2\sin\left(\dfrac{u+v}{2}\right)\cos\left(\dfrac{u-v}{2}\right)=\sin(u)+\sin(v)$ Establishing the identity

Example 9

Evaluate $\cos(15°)-\cos(75°)$

Using the sum to produce identity for the difference of cosines,

$\cos(15°)-\cos(75°)$

$=-2\sin\left(\dfrac{15°+75°}{2}\right)\sin\left(\dfrac{15°-75°}{2}\right)$ Simplify

$=-2\sin(45°)\sin(-30°)$ Evaluate

$=-2\cdot\dfrac{\sqrt{2}}{2}\cdot\dfrac{-1}{2}=\dfrac{\sqrt{2}}{2}$

Example 10

Prove the identity $\dfrac{\cos(4t)-\cos(2t)}{\sin(4t)+\sin(2t)}=-\tan(t)$

Since the left side seems more complicated, we can start there and simplify.

$\dfrac{\cos(4t)-\cos(2t)}{\sin(4t)+\sin(2t)}$ Using the sum to product identities

$=\dfrac{-2\sin\left(\dfrac{4t+2t}{2}\right)\sin\left(\dfrac{4t-2t}{2}\right)}{2\sin\left(\dfrac{4t+2t}{2}\right)\cos\left(\dfrac{4t-2t}{2}\right)}$ Simplify

$=\dfrac{-2\sin(3t)\sin(t)}{2\sin(3t)\cos(t)}$ Simplify further

$=\dfrac{-\sin(t)}{\cos(t)}$ Rewrite as a tangent

$=-\tan(t)$ Establishing the identity

Try it Now

5. Notice that using the negative angle identity, $\sin(u)-\sin(v)=\sin(u)+\sin(-v)$. Use this along with the sum of sines identity to prove the sum to product identity for $\sin(u)-\sin(v)$.

Example 11

Solve $\sin(\pi t)+\sin(3\pi t)=\cos(\pi t)$ for all solutions $0 \le t < 2$

In an equation like this, it is not immediately obvious how to proceed. One option would be to combine the two sine functions on the left side of the equation. Another would be to move the cosine to the left side of the equation, and combine it with one of the sines. For no particularly good reason, we'll begin by combining the sines on the left side of the equation and see how things work out.

$\sin(\pi t)+\sin(3\pi t)=\cos(\pi t)$	Apply the sum to product identity on the left
$2\sin\left(\frac{\pi t+3\pi t}{2}\right)\cos\left(\frac{\pi t-3\pi t}{2}\right)=\cos(\pi t)$	Simplify
$2\sin(2\pi t)\cos(-\pi t)=\cos(\pi t)$	Apply the negative angle identity
$2\sin(2\pi t)\cos(\pi t)=\cos(\pi t)$	Rearrange the equation to be = 0
$2\sin(2\pi t)\cos(\pi t)-\cos(\pi t)=0$	Factor out the cosine
$\cos(\pi t)\left(2\sin(2\pi t)-1\right)=0$	

Using the zero product theorem we know we will have solutions if either factor is zero. With the first part, $\cos(\pi t)=0$, the cosine has period $P=\frac{2\pi}{\pi}=2$, so the solution interval of $0 \le t < 2$ contains one full cycle of this function.

$\cos(\pi t)=0$	Substitute $u=\pi t$
$\cos(u)=0$	On one cycle, this has solutions
$u=\frac{\pi}{2}$ or $u=\frac{3\pi}{2}$	Undo the substitution

$\pi t=\frac{\pi}{2}$, so $t=\frac{1}{2}$

$\pi t=\frac{3\pi}{2}$, so $t=\frac{3}{2}$

For the second part of the equation, $2\sin(2\pi t)-1=0$, the sine has a period of $P=\frac{2\pi}{2\pi}=1$, so the solution interval $0 \le t < 2$ contains two cycles of this function.

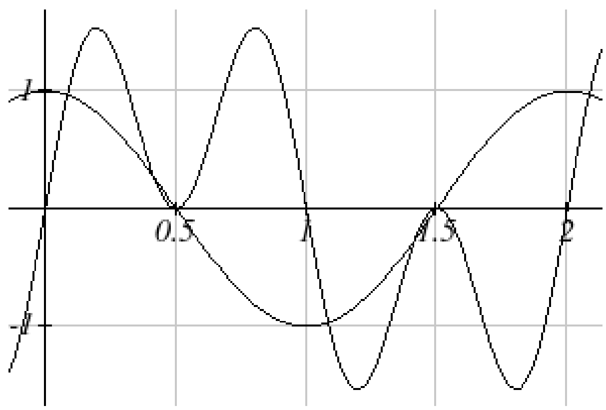
$2\sin(2\pi t) - 1 = 0$	Isolate the sine
$\sin(2\pi t) = \frac{1}{2}$	$u = 2\pi t$
$\sin(u) = \frac{1}{2}$	On one cycle, this has solutions
$u = \frac{\pi}{6}$ or $u = \frac{5\pi}{6}$	On the second cycle, the solutions are
$u = 2\pi + \frac{\pi}{6} = \frac{13\pi}{6}$ or $u = 2\pi + \frac{5\pi}{6} = \frac{17\pi}{6}$	Undo the substitution

$2\pi t = \frac{\pi}{6}$, so $t = \frac{1}{12}$

$2\pi t = \frac{5\pi}{6}$, so $t = \frac{5}{12}$

$2\pi t = \frac{13\pi}{6}$, so $t = \frac{13}{12}$

$2\pi t = \frac{17\pi}{6}$, so $t = \frac{17}{12}$



Altogether, we found six solutions on $0 \le t < 2$, which we can confirm as all solutions looking at the graph.

$t = \frac{1}{12}, \frac{5}{12}, \frac{1}{2}, \frac{13}{12}, \frac{3}{2}, \frac{17}{12}$

Important Topics of This Section

The sum and difference identities
Combining waves of equal periods
Product to sum identities
Sum to product identities
Completing proofs

Try it Now Answers

1.
$$\cos(\alpha + \beta) = \cos(\alpha - (-\beta))$$
$$\cos(\alpha)\cos(-\beta) + \sin(\alpha)\sin(-\beta)$$
$$\cos(\alpha)\cos(\beta) + \sin(\alpha)(-\sin(\beta))$$
$$\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

2. $\dfrac{\sqrt{2}-\sqrt{6}}{4}$

3. $6\sin\left(5x+\dfrac{3\pi}{4}\right)$

4. $\dfrac{-2+\sqrt{3}}{4}$

5. $\sin(u)-\sin(v)$ Use negative angle identity for sine

$\sin(u)+\sin(-v)$ Use sum to product identity for sine

$2\sin\left(\dfrac{u+(-v)}{2}\right)\cos\left(\dfrac{u-(-v)}{2}\right)$ Eliminate the parenthesis

$2\sin\left(\dfrac{u-v}{2}\right)\cos\left(\dfrac{u+v}{2}\right)$ Establishing the identity

Section 7.3 Double Angle Identities

While the sum of angles identities can handle a wide variety of cases, the double angle cases come up often enough that we choose to state these identities separately. The double angle identities are just another form of the sum of angle identities, since $\sin(2\alpha) = \sin(\alpha + \alpha)$.

Identities

The double angle identities

$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$

$$\begin{aligned}\cos(2\alpha) &= \cos^2(\alpha) - \sin^2(\alpha)\\ &= 1 - 2\sin^2(\alpha)\\ &= 2\cos^2(\alpha) - 1\end{aligned}$$

These identities follow from the sum of angles identities.

<u>Proof of the sine double angle identity</u>

$\sin(2\alpha)$	
$= \sin(\alpha + \alpha)$	Apply the sum of angles identity
$= \sin(\alpha)\cos(\alpha) + \cos(\alpha)\sin(\alpha)$	Simplify
$= 2\sin(\alpha)\cos(\alpha)$	Establishing the identity

Try it Now

1. Show $\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha)$ by using the sum of angles identity for cosine

For the cosine double angle identity, there are three forms of the identity that are given because the basic form, $\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha)$, can be rewritten using the Pythagorean Identity. Rearranging the Pythagorean Identity results in the equality $\cos^2(\alpha) = 1 - \sin^2(\alpha)$, and by substituting this into the basic double angle identity, we obtain the second form of the double angle identity.

$\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha)$	Substituting using the Pythagorean identity
$\cos(2\alpha) = 1 - \sin^2(\alpha) - \sin^2(\alpha)$	Simplifying
$\cos(2\alpha) = 1 - 2\sin^2(\alpha)$	

Example 1

If $\sin(\theta)=\dfrac{3}{5}$ and θ is in the second quadrant, find exact values for $\sin(2\theta)$ and $\cos(2\theta)$

To evaluate $\cos(2\theta)$, since we know the value for the sine, we can use the version of the double angle that only involves sine.

$$\cos(2\theta)=1-2\sin^2(\theta)=1-2\left(\frac{3}{5}\right)^2=1-\frac{18}{5}=-\frac{8}{5}$$

Since the double angle for sine involves both sine and cosine, we'll need to first find $\cos(\theta)$, which we can do using the Pythagorean identity.

$$\sin^2(\theta)+\cos^2(\theta)=1$$

$$\left(\frac{3}{5}\right)^2+\cos^2(\theta)=1$$

$$\cos^2(\theta)=1-\frac{9}{25}$$

$$\cos(\theta)=\pm\sqrt{\frac{16}{25}}=\pm\frac{4}{5}$$

Since θ is in the second quadrant, we want to keep the negative value for cosine,

$$\cos(\theta)=-\frac{4}{5}$$

Now we can evaluate the sine double angle

$$\sin(2\theta)=2\sin(\theta)\cos(\theta)=2\left(\frac{3}{5}\right)\left(-\frac{4}{5}\right)=-\frac{24}{25}$$

We can use the double angle identities for simplifying expressions and proving identities.

Example 2

Simplify $\dfrac{\cos(2t)}{\cos(t)-\sin(t)}$

With three choices for how to rewrite the double angle, we need to consider which will be the most useful. To simplify this expression, it would be great if the fraction would cancel, which would require a factor of $\cos(t)-\sin(t)$, which is most likely to occur if we rewrite the numerator with a mix of sine and cosine.

$\dfrac{\cos(2t)}{\cos(t)-\sin(t)}$	Apply the double angle identity
$=\dfrac{\cos^2(t)-\sin^2(t)}{\cos(t)-\sin(t)}$	Factor the numerator
$=\dfrac{\left(\cos(t)-\sin(t)\right)\left(\cos(t)+\sin(t)\right)}{\cos(t)-\sin(t)}$	Cancelling the common factor
$=\cos(t)+\sin(t)$	Resulting in the most simplified form

Example 3

Prove $\sec(2\alpha)=\dfrac{\sec^2(\alpha)}{2-\sec^2(\alpha)}$

Since the right side seems a bit more complex than the left side, we begin there.

$\dfrac{\sec^2(\alpha)}{2-\sec^2(\alpha)}$	Rewrite the secants in terms of cosine
$=\dfrac{\dfrac{1}{\cos^2(\alpha)}}{2-\dfrac{1}{\cos^2(\alpha)}}$	Find a common denominator on the bottom
$=\dfrac{\dfrac{1}{\cos^2(\alpha)}}{\dfrac{2\cos^2(\alpha)}{\cos^2(\alpha)}-\dfrac{1}{\cos^2(\alpha)}}$	Subtract the terms in the denominator
$=\dfrac{\dfrac{1}{\cos^2(\alpha)}}{\dfrac{2\cos^2(\alpha)-1}{\cos^2(\alpha)}}$	Invert and multiply
$=\dfrac{1}{\cos^2(\alpha)}\cdot\dfrac{\cos^2(\alpha)}{2\cos^2(\alpha)-1}$	Cancel the common factors
$=\dfrac{1}{2\cos^2(\alpha)-1}$	Rewrite the denominator as a double angle
$=\dfrac{1}{\cos(2\alpha)}$	Rewrite as a secant
$=\sec(2\alpha)$	Establishing the identity

Try it Now

2. Use an identity to find the exact value of $\cos^2(75°)-\sin(75°)$

Like with other identities, we can also use the double angle identities for solving equations.

Example 4

Solve $\cos(2t) = \cos(t)$ for all solutions $0 \le t < 2\pi$

In general when solving trig equations, it makes things more complicated when we have a mix of sines and cosines and when we have a mix of functions with different periods. In this case, we can use a double angle identity to rewrite the double angle term. When choosing which form of the double angle identity to use, we notice that we have a cosine on the right side of the equation. We try to limit our equation to one trig function, which we can do by choosing the version of the double angle that only involves cosine.

$\cos(2t) = \cos(t)$	Apply the double angle identity
$2\cos^2(t) - 1 = \cos(t)$	This is quadratic in cosine, so rearrange it = 0
$2\cos^2(t) - \cos(t) - 1 = 0$	Factor
$(2\cos(t) + 1)(\cos(t) - 1) = 0$	Break this apart to solve each part separately

$2\cos(t) + 1 = 0$	or	$\cos(t) - 1 = 0$
$\cos(t) = -\dfrac{1}{2}$	or	$\cos(t) = 1$
$t = \dfrac{2\pi}{3}$ or $t = \dfrac{4\pi}{3}$	or	$t = 0$

Example 5

A cannonball is fired with velocity of 100 meters per second. If it is launched at an angle of θ, the vertical component of the velocity will be $100\sin(\theta)$ and the horizontal component will be $100\cos(\theta)$. Ignoring wind resistance, the height of the cannonball will follow the equation $h(t) = -4.9t^2 + 100\sin(\theta)t$ and horizontal position will follow the equation $x(t) = 100\cos(\theta)t$. If you want to hit a target 900 meters away, at what angle should you aim the cannon?

To hit the target 20 miles away, we want $x(t) = 900$ at the time when the cannonball hits the ground, when $h(t) = 0$. To solve this problem, we will first solve for the time, t, when the cannonball hits the ground. Our answer will depend upon the angle θ.

$h(t) = 0$	
$-4.9t^2 + 100\sin(\theta)t = 0$	Factor
$t(-4.9t + 100\sin(\theta)) = 0$	Break this apart to find two solutions

$t = 0$ or
$-4.9t + 100\sin(\theta) = 0$ Solve for t
$-4.9t = -100\sin(\theta)$
$t = \dfrac{100\sin(\theta)}{4.9}$

This shows that the height is 0 twice, once at $t = 0$ when the ball is first thrown, and again when the ball hits the ground. The second value of t gives the time when the ball hits the ground as a function of the angle θ. We want the horizontal distance *x(t)* to be 12 when the ball hits the ground , so when $t = \dfrac{100\sin(\theta)}{4.9}$.

Since the target is 900 m away we start with

$x(t) = 900$ Use the formula for *x(t)*

$100\cos(\theta)t = 900$ Substitute the desired time, t from above
$100\cos(\theta)\dfrac{100\sin(\theta)}{4.9} = 900$ Simplify
$\dfrac{100^2}{4.9}\cos(\theta)\sin(\theta) = 900$ Isolate the cosine and sine product
$\cos(\theta)\sin(\theta) = \dfrac{900(4.9)}{100^2}$

The left side of this equation almost looks like the result of the double angle identity for sine: $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$.

By dividing both sides of the double angle identity by 2, we get
$\dfrac{1}{2}\sin(2\alpha) = \sin(\alpha)\cos(\alpha)$. Applying this to the equation above,

$\dfrac{1}{2}\sin(2\theta) = \dfrac{900(4.9)}{100^2}$ Multiply by 2
$\sin(2\theta) = \dfrac{2(900)(4.9)}{100^2}$ Use the inverse
$2\theta = \sin^{-1}\left(\dfrac{2(900)(4.9)}{100^2}\right) \approx 1.080$ Divide by 2
$\theta = \dfrac{1.080}{2} = 0.540$, or about 30.94 degrees

Power Reduction and Half Angle Identities

Another use of the cosine double-angle identities is to use them in reverse to rewrite a squared sine or cosine in terms of the double angle. Starting with one form of the cosine double angle identity:

$\cos(2\alpha) = 2\cos^2(\alpha) - 1$ — Isolate the cosine squared

$\cos(2\alpha) + 1 = 2\cos^2(\alpha)$ — Adding 1

$\cos^2(\alpha) = \dfrac{\cos(2\alpha)+1}{2}$ — Dividing by 2

$\cos^2(\alpha) = \dfrac{\cos(2\alpha)+1}{2}$ — This is called a **power reduction identity**

Try it Now

3. Use another form of the cosine double angle identity to prove the identity $\sin^2(\alpha) = \dfrac{1-\cos(2\alpha)}{2}$

Example 6

Rewrite $\cos^4(x)$ without any powers

Since $\cos^4(x) = \left(\cos^2(x)\right)^2$, we can use the formula we found above

$\cos^4(x) = \left(\cos^2(x)\right)^2$

$= \left(\dfrac{\cos(2x)+1}{2}\right)^2$ — Square the numerator and denominator

$= \dfrac{\left(\cos(2x)+1\right)^2}{4}$ — FOIL the top & square the bottom

$= \dfrac{\cos^2(2x)+2\cos(2x)+1}{4}$ — Split apart the fraction

$= \dfrac{\cos^2(2x)}{4} + \dfrac{2\cos(2x)}{4} + \dfrac{1}{4}$ — Apply the formula above to $\cos^2(2x)$

$= \dfrac{\left(\dfrac{\cos(4x)+1}{2}\right)}{4} + \dfrac{2\cos(2x)}{4} + \dfrac{1}{4}$ — Simplify

$= \dfrac{\cos(4x)}{8} + \dfrac{1}{8} + \dfrac{1}{2}\cos(2x) + \dfrac{1}{2}$ — Combine the constants

$= \dfrac{\cos(4x)}{8} + \dfrac{1}{2}\cos(2x) + \dfrac{5}{8}$

The cosine double angle identities can also be used in reverse for evaluating angles that are half of a common angle. Building off our formula $\cos^2(\alpha) = \dfrac{\cos(2\alpha)+1}{2}$ from earlier, if we let $\theta = 2\alpha$, then this identity becomes $\cos^2\left(\dfrac{\theta}{2}\right) = \dfrac{\cos(\theta)+1}{2}$. Taking the square root, we obtain

$\cos\left(\dfrac{\theta}{2}\right) = \pm\sqrt{\dfrac{\cos(\theta)-1}{2}}$, where the sign is determined by the quadrant.

This is called a **half-angle identity**.

Try it Now

4. Use your results from the last Try it Now to prove the identity

$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{2}}$$

Example 7

Find an exact value for $\cos(15°)$.

Since 15 degrees is half of 30 degrees, we can use our result from above:

$$\cos(15°) = \cos\left(\frac{30°}{2}\right) = \pm\sqrt{\frac{\cos(30°)-1}{2}}$$

We can evaluate the cosine. Since 15 degrees is in the first quadrant, we will keep the positive result.

$$\sqrt{\frac{\cos(30°)-1}{2}} = \sqrt{\frac{\frac{\sqrt{3}}{2}-1}{2}}$$

$$= \sqrt{\frac{\sqrt{3}}{4}-\frac{1}{2}}$$

Identities

Half-Angle Identities

$$\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{\cos(\theta)-1}{2}} \qquad \sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{2}}$$

Power Reduction Identities

$$\cos^2(\alpha) = \frac{\cos(2\alpha)+1}{2} \qquad \sin^2(\alpha) = \frac{1-\cos(2\alpha)}{2}$$

Since these identities are easy to derive from the double-angle identities, the power reduction and half-angle identities are not ones you should need to memorize separately.

Important Topics of This Section

Double angle identity
Power reduction identity
Half angle identity
Using identities
- Simplify equations
- Prove identities
- Solve equations

Try it Now Answers

1. $$\cos(2\alpha) = \cos(\alpha + \alpha)$$
$$\cos(\alpha)\cos(\alpha) - \sin(\alpha)\sin(\alpha)$$
$$\cos^2(\alpha) - \sin^2(\alpha)$$

2. $\cos(150°) = \dfrac{-\sqrt{3}}{2}$

3. $$\frac{1-\cos(2\alpha)}{2}$$
$$\frac{1-\left(\cos^2(\alpha)-\sin^2(\alpha)\right)}{2}$$
$$\frac{1-\cos^2(\alpha)+\sin^2(\alpha)}{2}$$
$$\frac{\sin^2(\alpha)+\sin^2(\alpha)}{2}$$
$$\frac{2\sin^2(\alpha)}{2} = \sin^2(\alpha)$$

$$\sin^2(\alpha) = \frac{1-\cos(2\alpha)}{2}$$

$$\sin(\alpha) = \pm\sqrt{\frac{1-\cos(2\alpha)}{2}}$$

4. $\alpha = \dfrac{\theta}{2}$

$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\left(2\left(\frac{\theta}{2}\right)\right)}{2}}$$

$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{2}}$$

Section 7.4 Modeling Changing Amplitude and Midline

While sinusoidal functions can model a variety of behaviors, often it is necessary to combine sinusoidal functions with linear and exponential curves to model real applications and behaviors. We begin this section by looking at changes to the midline of a sinusoidal function. Recall that the midline describes the middle, or average value, of the sinusoidal function.

Changing Midlines

Example 1

A population of elk currently averages 2000 elk, and that average has been growing by 4% each year. Due to seasonal fluctuation, the population oscillates 50 below average in the winter up to 50 above average in the summer. Write an equation for the number of elk after *t* years.

There are two components to the behavior of the elk population: the changing average, and the oscillation. The average is an exponential growth, starting at 2000 and growing by 4% each year. Writing a formula for this:

$$average = initial(1+r)^t = 2000(1+0.04)^t$$

For the oscillation, since the population oscillates 50 above and below average, the amplitude will be 50. Since it takes one year for the population to cycle, the period is 1. We find the value of the horizontal stretch coefficient $B = \dfrac{original\ period}{new\ period} = \dfrac{2\pi}{1} = 2\pi$.

Additionally, since we weren't told when *t* was first measured we will have to decide if $t = 0$ corresponds to winter, or summer. If we choose winter then the shape of the function would be a negative cosine, since it starts at the lowest value.

Putting it all together, the equation would be:

$$P(t) = -50\cos(2\pi t) + midline$$

Since the midline represents the average population, we substitute in the exponential function into the population equation to find our final equation:

$$P(t) = -50\cos(2\pi t) + 2000(1+0.04)^t$$

This is an example of changing midline – in this case an exponentially changing midline.

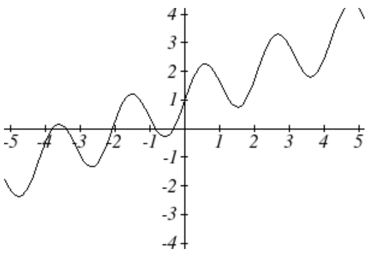
Definition

Changing Midline

A function of the form $f(t) = A\sin(Bt) + g(t)$ will oscillate above and below the average given by the function *g(t)*.

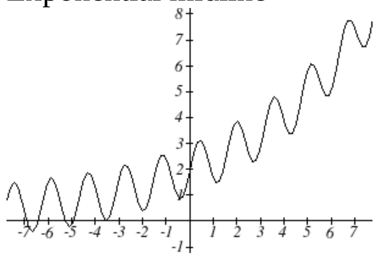
Changing midlines can be exponential, linear, or any other type of function. Here are some examples of what the resulting functions would look like.

Linear midline



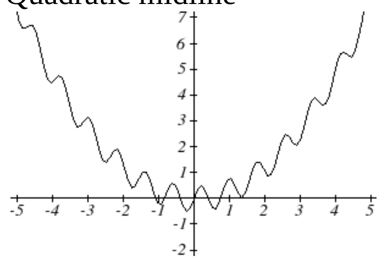
$f(t) = A\sin(Bt) + (mt + b)$

Exponential midline



$f(t) = A\sin(Bt) + (ab^t)$

Quadratic midline



$f(t) = A\sin(Bt) + (at^2)$

Example 2

Find a function with linear midline of the form $f(t) = A\sin\left(\frac{\pi}{2}t\right) + mt + b$ that will pass through points below.

t	0	1	2	3
f(t)	5	10	9	8

Since we are given the value of the horizontal compression coefficient we can calculate the period of this function: $new\ period = \frac{original\ period}{B} = \frac{2\pi}{\pi/2} = 4$.

Since the sine function is at the midline at the beginning of a cycle and halfway through a cycle, we would expect this function to be at the midline at $t = 0$ and $t = 2$, since 2 is half the full period of 4. Based on this, we expect the points (0, 5) and (2, 9) to be points on the midline. We can clearly see that this is not a constant function and so we use the two points to calculate a linear function: $midline = mt + b$. From these two points we can calculate a slope:

$$m = \frac{9-5}{2-0} = \frac{4}{2} = 2$$

Combining this with the initial value of 5, we have the midline: $midline = 2t + 5$, giving a full function of the form $f(t) = A\sin\left(\frac{\pi}{2}t\right) + 2t + 5$. To find the amplitude, we can plug in a point we haven't already used, such as (1, 10)

$10 = A\sin\left(\frac{\pi}{2}(1)\right) + 2(1) + 5$ Evaluate the sine and combine like terms

$10 = A + 7$

$A = 3$

An equation of the form given fitting the data would be

$$f(t) = 3\sin\left(\frac{\pi}{2}t\right) + 2t + 5$$

Alternative Approach

Notice we could have taken an alternate approach by plugging points (0, 5) and (2, 9) into the original equation. Substituting (0, 5),

$5 = A\sin\left(\frac{\pi}{2}(0)\right) + m(0) + b$ Evaluate the sine and simplify

$5 = b$

Substituting (2, 9)

$9 = A\sin\left(\frac{\pi}{2}(2)\right) + m(2) + 5$ Evaluate the sine and simplify

$9 = 2m + 5$

$4 = 2m$

$m = 2$, as we found above.

Example 3

The number of tourists visiting a ski and hiking resort averages 4000 people annually and oscillates seasonally, 1000 above and below the average. Due to a marketing campaign, the average number of tourists has been increasing by 200 each year. Write an equation for the number of tourists *t* years, beginning at the peak season.

Again there are two components to this problem: the oscillation and the average. For the oscillation, the number oscillates 1000 above and below average, giving an amplitude of 1000. Since the oscillation is seasonal, it has a period of 1 year. Since we are given a starting point of “peak season”, we will model this scenario with a cosine function.

So far, this gives an equation in the form $N(t) = 1000\cos(2\pi t) + midline$

For the average, the average is currently 4000, and is increasing by 200 each year. This is a constant rate of change, so this is linear growth, $average = 4000 + 200t$.

Combining these two pieces gives an equation for the number of tourists:

$N(t) = 1000\cos(2\pi t) + 4000 + 200t$

Try it Now

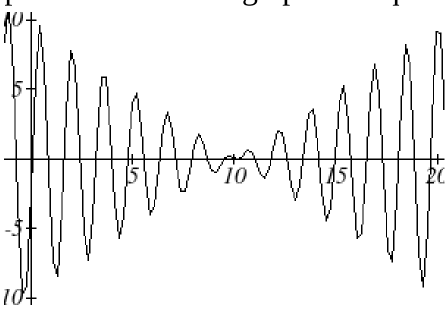
1. Given the function $g(x) = (x^2 - 1) + 8\cos(x)$ describe the midline and amplitude in words.

Changing Amplitude
As with midline, there are times when the amplitude of a sinusoidal function does not stay constant. Back in chapter 6, we modeled the motion of a spring using a sinusoidal function, but had to ignore friction in doing so. If there were friction in the system, we would expect the amplitude of the oscillation to decrease over time. Since in the equation $f(t) = A\sin(Bt) + k$, *A* gives the amplitude of the oscillation, we can allow the amplitude to change by changing this constant *A* to a function *A(t)*.

Definition

Changing Amplitude
A function of the form $f(t) = A(t)\sin(Bt) + k$ will oscillate above and below the midline with an amplitude given by *A(t)*.

When thinking about a spring with amplitude decreasing over time, it is tempting to use the simplest equation for the job – a linear function. But if we attempt to model the amplitude with a decreasing linear function, such as $A(t) = 10 - t$, we quickly see the problem when we graph the equation $f(t) = (10 - t)\sin(4t)$.



While the amplitude decreases at first as intended, the amplitude hits zero at *t* = 10, then continues past the intercept, increasing in absolute value, which is not the expected behavior. This behavior and function may model the situation well on a restricted domain and we might try to chalk the rest of it up to model breakdown, but in fact springs just don't behave like this.

A better model would show the amplitude decreasing by a *percent* each second, leading to an exponential decay model for the amplitude.

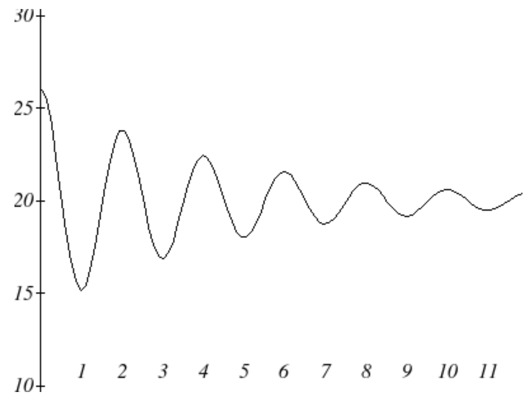
Definition

Damped harmonic motion, exhibited by springs subject to friction, follows an equation of the form
$f(t) = ab^t\sin(Bt) + k$ or $f(t) = ae^{rt}\sin(Bt) + k$ for continuous decay.

Example 4

A spring with natural length of 20 inches is pulled back 6 inches and released. It oscillates once every 2 seconds. Its amplitude decreases by 20% each second. Write an equation for the position of the spring t seconds after being released.

Since the spring will oscillate on either side of the natural length, the midline will be at 20 inches. The oscillation has a period of 2 seconds, and so the horizontal compression coefficient is $B = \pi$. Additionally, it begins at the furthest distance from the wall, indicating a cosine model.



Meanwhile, the amplitude begins at 6 inches, and decreases by 20% each second, giving an amplitude equation of $A(t) = 6(1 - 0.20)^t$.

Combining this with the sinusoidal information gives an equation for the position of the spring:
$f(t) = 6(0.80)^t \cos(\pi t) + 20$

Example 5

A spring with natural length of 30 cm is pulled out 10 cm and released. It oscillates 4 times per second. After 2 seconds, the amplitude has decreased to 5 cm. Find an equation for the position of the spring.

The oscillation has a period of ¼ second. Since the spring will oscillate on either side of the natural length, the midline will be at 30 cm. It begins at the furthest distance from the wall, suggesting a cosine model. Together, this gives
$f(t) = A(t)\cos(8\pi t) + 30$

For the amplitude function, we notice that the amplitude starts at 10 cm, and decreased to 5 cm after 2 seconds. This gives two points (0, 10) and (2, 5) that must be satisfied by the exponential equation: $A(0) = 10$ and $A(2) = 5$. Since the equation is exponential, we can use the form $A(t) = ab^t$. Substituting the first point, $10 = ab^0$, so a = 10. Substituting in the second point,

$5 = 10b^2$	Divide by 10
$\frac{1}{2} = b^2$	Take the square root
$b = \sqrt{\frac{1}{2}} \approx 0.707$	

This gives an amplitude equation of $A(t) = 10(0.707)^t$. Combining this with the oscillation,
$f(t) = 10(0.707)^t \cos(8\pi t) + 30$

Try it Now

2. A certain stock started at a high value of \$7 per share and has been oscillating above and below the average value, decreasing by 2% per year. However, the average value started at \$4 per share and has grown linearly by 50 cents per year.

   a. Write an equation for the midline
   b. Write an equation for the amplitude.
   c. Find the equation S(t) for the value of the stock after t years.

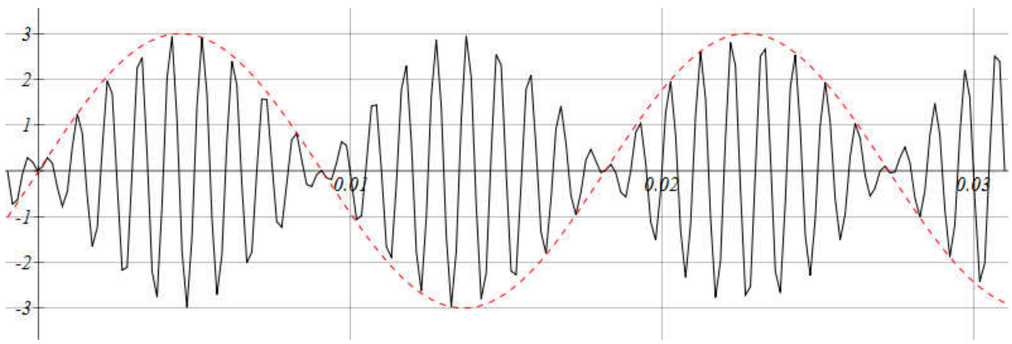
Example 6

In Amplitude Modulated (AM) radio, a carrier wave with a high frequency is used to transmit music or other signals by applying the transmit signal as the amplitude of the carrier signal. A musical note with frequency 110 Hz (Hertz - cycles per second) is to be carried on a wave with frequency of 2 KHz (Kilohertz – thousands of cycles per second). If a musical wave has an amplitude of 3, write an equation describing the broadcast wave.

The carrier wave, with a frequency of 2000 cycles per second, would have period $\frac{1}{2000}$ of a second, giving an equation of the form $\sin(4000\pi t)$. Our choice of a sine function here was arbitrary – it would have worked just was well to use a cosine.

For the music note, with a frequency of 110 cycles per second, it would have a period of $\frac{1}{110}$ of a second. With an amplitude of 3, this would have an equation of the form $3\sin(220\pi t)$. Again our choice of using a sine function is arbitrary.

The musical wave is acting as the amplitude of the carrier wave, so we will multiply the music wave function with the carrier wave function, giving a resulting equation $f(t) = 3\sin(220\pi t)\sin(4000\pi t)$



Important Topics of This Section

Changing midline
Changing amplitude
- Linear Changes
- Exponential Changes
- Damped Harmonic Motion

Try it Now Answers

1. The midline follows the path of the quadratic $x^2 - 1$ and the amplitude is a constant value of 8.

2. $m(t) = 4 + 0.5t$
$A(t) = 7(0.8)^t$
$S(t) = 7(0.8)^t \cos\left(\frac{\pi}{6}t\right) + 4 + 0.5t$